



Renormalisation group, trace anomaly and Feynman–Hellmann theorem

Luigi Del Debbio, Roman Zwicky*

Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland, United Kingdom

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ABSTRACT

We show that the logarithmic derivative of the gauge coupling on the hadronic mass and the cosmological constant term of a gauge theory are related to the gluon condensate of the hadron and the vacuum respectively. These relations are akin to Feynman–Hellmann relations whose derivation for the case at hand is complicated by the construction of the gauge theory Hamiltonian. We bypass this problem by using a renormalisation group equation for composite operators and the trace anomaly. The relations serve as possible definitions of the gluon condensates themselves which are plagued in direct approaches by power divergences. In turn these results might help to determine the contribution of the QCD phase transition to the cosmological constant and test speculative ideas.

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1. Introduction

The Feynman–Hellmann theorem relates the leading order variation of the energy to a local matrix element, providing a direct link between an observable and a theoretical quantity. Originally derived in quantum mechanics, its application to quantum field theory (QFT) is generally straightforward and widely used, see e.g. [1]. Exceptions are cases where the Hamiltonian is difficult to construct, which may arise as a QFT is usually defined from a Lagrangian where (most) symmetries are manifest. An example of such a case are gauge theories where the elimination of two degrees of freedom from the four vector A_μ is at the root of the problem. In this work we bypass the construction of the gauge field part of the Hamiltonian¹ by using renormalisation group equations (RGE) for composite operators as well as the trace anomaly. We obtain a relation that relates the logarithmic derivative of the hadron mass with respect to the coupling constant, and the gluon condensate of the hadron state. Likewise we find a similar relation relating the derivative of the cosmological constant and the vacuum gluon condensate.

The corresponding relation for the quark mass was used in Ref. [3] to derive the leading scaling behaviour of the hadronic masses for a non-trivial infrared (IR) fixed point, deformed by the

fermion mass parameter. The relation derived in this paper is used to compute the scaling corrections to the hadronic mass spectrum [4].

2. Preliminary results

We shall first rederive some results before assembling them to obtain the main relations of this paper.

2.1. RGE for matrix elements of local composite operators

In this section we outline the derivation of the standard RGE for local operator matrix elements on physical states which can be found in reference textbooks; e.g. [5,6]. We begin by defining the relation between the bare operator O_i and the renormalised operator \tilde{O}_i

$$O_i(g, m, \Lambda) = (\tilde{Z}_O)_{ij}^{-1}(\mu/\Lambda) \tilde{O}_j(\tilde{g}, \tilde{m}, \mu), \quad (1)$$

where summation over indices is implied, (g, m) are the bare gauge coupling and mass, Λ is the UV cut-off of the theory, (\tilde{g}, \tilde{m}) are the set of renormalised couplings and μ is the renormalisation scale. As stated above the operators are understood to be evaluated between two physical states in order to avoid the issue of contact terms which arises upon insertion of additional operators. From the independence of the bare operator on the renormalisation scale,

$$\frac{d}{d \ln \mu} O_i(g, m, \Lambda) = 0, \quad (2)$$

* Corresponding author.

E-mail addresses: luigi.del.debbio@ed.ac.uk (L. Del Debbio), roman.zwicky@ed.ac.uk (R. Zwicky).

¹ In subsequent work the relations, obtained in this paper, have been derived using the Hamiltonian formalism [2].

one obtains an RGE of the form,

$$\left(\frac{\partial}{\partial \ln \mu} + \bar{\beta} \frac{\partial}{\partial \bar{g}} - \bar{m} \bar{\gamma}_m \frac{\partial}{\partial \bar{m}} + (\bar{\gamma}_O)_{ij} \right) \bar{O}_j(\bar{g}, \bar{m}, \mu) = 0 \quad (3)$$

with

$$\bar{\beta} \equiv \frac{d\bar{g}}{d \ln \mu}, \quad \bar{\gamma}_m \equiv -\frac{d \ln \bar{m}}{d \ln \mu},$$

$$(\bar{\gamma}_O)_{ij} \equiv -(\bar{Z}_O^{-1})_{ik} \frac{d(\bar{Z}_O)_{kj}}{d \ln \mu}. \quad (4)$$

Denoting by $d_{O_i} \equiv d_{\bar{O}_i}$ the engineering dimension of O_i one gets by dimensional analysis

$$\left(\frac{\partial}{\partial \ln \mu} + \bar{m} \frac{\partial}{\partial \bar{m}} - d_{O_i} \right) \bar{O}_i(\bar{g}, \bar{m}, \mu) = 0, \quad (5)$$

an equation which can be combined with (3) into

$$\left(\bar{\beta} \frac{\partial}{\partial \bar{g}} - (1 + \bar{\gamma}_m) \bar{m} \frac{\partial}{\partial \bar{m}} + (\bar{\Delta}_O)_{ij} \right) \bar{O}_j(\bar{g}, \bar{m}, \mu) = 0. \quad (6)$$

Eq. (6) is an RGE equation for the composite operator, sometimes referred to as the 't Hooft–Weinberg or Callan–Symanzik equation. The symbol $\bar{\Delta}_O \equiv d_O + \bar{\gamma}_O$ is, as usual, the scaling dimension of the operator \bar{O} . Eq. (6) can be solved by the method of characteristics by introducing a parameter which has the interpretation of a blocking variable. This is for instance used in Ref. [4] to identify the scaling corrections to correlators at a non-trivial IR fixed point.

To this end we note that in this paper the O_i considered are physical quantities (no anomalous scaling) which in addition do not mix with other operators and therefore $(\Delta_O)_{ij} = d_{O_i} \delta_{ij}$.

2.2. Trace anomaly

Let us first define some conventions. The Lorentz-invariant normalisation of states for D -dimensional space–time is given by

$$\langle H(E', \vec{p}') | H(E, \vec{p}) \rangle = 2E(\vec{p})(2\pi)^{D-1} \delta^{(D-1)}(\vec{p} - \vec{p}'), \quad (7)$$

the diagonal matrix elements are abbreviated to

$$\langle X \rangle_{E_H} \equiv \langle H(E, \vec{p}) | X | H(E, \vec{p}) \rangle_c, \quad (8)$$

where c denotes the connected part and H stands for any physical state. Since the energy momentum tensor is related to the four momentum operator $\hat{p}_\mu = \int d^{D-1}x T_{0\mu}$ it is readily seen that:

$$\langle T_{\mu\nu} \rangle_{E_H} = 2p_\mu p_\nu, \quad \langle T_{\mu\nu} \rangle_0 = \Lambda_{\text{GT}} g_{\mu\nu}, \quad (9)$$

where $\langle O \rangle_0 \equiv \langle 0 | O | 0 \rangle$ hereafter with $|0\rangle$ denoting the vacuum state, p_μ is the four momentum associated with the physical state ($E = p_0$), $g_{\mu\nu}$ is the Minkowski metric with signature $(+, -, \dots, -)$ and Λ_{GT} is the cosmological constant contribution of the gauge theory under consideration.

The traces of the right hand side (RHS) are the masses of the hadrons $2M_H^2$ and the masses density of empty space $D\Lambda_{\text{GT}} = g_{\mu}^{\mu} \Lambda_{\text{GT}}$, and the left hand sides (LHS) follow from the trace anomaly. For a gauge theory with N_f Dirac quarks the trace anomaly, in terms of renormalised fields, is given by [7],²

$$T_{\mu}^{\mu}|_{\text{on-shell}} = \frac{\bar{\beta}}{2\bar{g}} \bar{G} + (1 + \bar{\gamma}_m) \bar{Q}, \quad (10)$$

² In terms of bare fields the expression (in dimensional regularisation) reads: $T_{\mu}^{\mu}|_{\text{on-shell}} = -(D-4)\frac{1}{4}G + Q$.

where the subscript “on-shell” (10) indicates that the equation, in this form, is to be used on the physical subspace only. Furthermore we have introduced the following shorthand notation:

$$Q \equiv N_f m \bar{q} q, \quad G \equiv \frac{1}{g^2} G_{\alpha\beta}^A G^{A\alpha\beta}, \quad (11)$$

with $G_{\alpha\beta}^A$ being the gauge field strength tensor and A a colour index. Summation over indices is understood. Conventions are such that the coupling is absorbed into the gauge field. We note that the trace energy momentum tensor is not renormalised (i.e. $T_{\mu\nu} = \bar{T}_{\mu\nu}$) as it is directly related to the four momentum which is a physical quantity. Furthermore $\bar{Q} = Q$ is an RG invariant which then implies that $\bar{\beta}/(2\bar{g})\bar{G} + \bar{\gamma}_m \bar{Q}$ is an RG invariant on the subspace of physical states. In particular we note $\bar{G} \neq G$ which is of importance when interpreting our final result. Finally Eqs. (10) and (9) lead to:

$$2M_H^2 = \frac{\bar{\beta}}{2\bar{g}} \bar{G}_{E_H} + (1 + \bar{\gamma}_m) \bar{Q}_{E_H} \quad (12)$$

$$D\Lambda_{\text{GT}} = \frac{\bar{\beta}}{2\bar{g}} \langle \bar{G} \rangle_0 + (1 + \bar{\gamma}_m) \langle \bar{Q} \rangle_0. \quad (13)$$

2.3. Feynman–Hellmann theorem in QFT

Let us start by recalling the main steps in the derivation of the Feynman–Hellmann theorem in quantum mechanics, which is a simple but powerful relation which has been obtained by a number of authors [8]. Let us consider a quantum-mechanical system, whose dynamics is determined by a Hamiltonian $\mathcal{H}(\lambda)$, which depends on some parameter λ . The Feynman–Hellmann theorem states that the λ -derivative of the energy equals the derivative of the Hamiltonian when evaluated on the corresponding eigenstates:

$$\frac{\partial}{\partial \lambda} E(\lambda) = \langle \Psi_{E(\lambda)} | \frac{\partial}{\partial \lambda} \mathcal{H}(\lambda) | \Psi_{E(\lambda)} \rangle. \quad (14)$$

It relies on the observation that

$$\langle \Psi_{E(\lambda)} | \Psi_{E(\lambda)} \rangle = 1 \Rightarrow \frac{\partial}{\partial \lambda} \langle \Psi_{E(\lambda)} | \Psi_{E(\lambda)} \rangle = 0. \quad (15)$$

The adaption to QFT, in the simplest cases, necessitates solely to take into account the relativistic state normalisation (7). E.g. with $\mathcal{H}_m = Q = \bar{Q}$ (11), where $\mathcal{H}_{\text{tot}} = \mathcal{H}_m + \dots$, the Feynman–Hellmann theorem for the mass reads:

$$\bar{m} \frac{\partial}{\partial \bar{m}} E_H^2 = \langle \bar{Q} \rangle_{E_H}, \quad (16)$$

$$\bar{m} \frac{\partial}{\partial \bar{m}} (D\Lambda_{\text{GT}}) = \langle \bar{Q} \rangle_0. \quad (17)$$

In (16) we have identified $(2\pi)^{D-1} \delta^{(D-1)}(\vec{p} - \vec{p}')_{\vec{p} \rightarrow \vec{p}'} \leftrightarrow \int d^{D-1}x$ in the sense of distributions. Note, the E_H^2 instead of E_H in (14) on the LHS originates from the additional factor of $2E_H$ in the normalisation (7). In (17) the normalisation $\langle 0 | 0 \rangle = 1$ was assumed. Furthermore we note that in a mass independent scheme ($\frac{\partial}{\partial \bar{m}} \bar{Z}_m = 0$) $\bar{m} \frac{\partial}{\partial \bar{m}} = m \frac{\partial}{\partial m}$ and since $Q = \bar{Q}$, therefore the relation (16) also holds for bare quantities. Eq. (16) is widely known [1] and used in lattice simulation to extract the corresponding contribution to the nucleon mass for example [11]. Noting that $m \frac{\partial}{\partial m} E_H^2 = m \frac{\partial}{\partial m} M_H^2$,³ it follows that $\langle Q \rangle_{E_H} = \langle Q \rangle_{M_H}$ with normalisation (7) is a static quantity.

³ We consider the states as used in (8) as momentum eigenstates and not boosted states and therefore \vec{p} has no dependence on M_H . More precisely in a lattice simulation the states originate from interpolating operators of the form: $\Phi(\vec{p}) = \int d^{D-1}x e^{i\vec{x} \cdot \vec{p}} \Phi(x_0, \vec{x})$.

3. Gluon condensates through RGE and Feynman–Hellmann theorem

The adaption of the analogous relations (16), (17) with regard to the gauge coupling g is complicated by the fact that in gauge theories the construction of the Hamiltonian itself is rather involved, see e.g. Ref. [9]. As stated in the introduction, we bypass the construction of the Hamiltonian, and its primary and secondary constraints, by using the RGE, the trace anomaly, and the relations for the mass.

The RGE (6) for the M_H^2 and $\Lambda_{\text{CT}}^{4,5}$

$$\left(\bar{\beta} \frac{\partial}{\partial \bar{g}} - (1 + \bar{\gamma}_m) \bar{m} \frac{\partial}{\partial \bar{m}} + 2 \right) M_H^2 = 0, \quad (18)$$

$$\left(\bar{\beta} \frac{\partial}{\partial \bar{g}} - (1 + \bar{\gamma}_m) \bar{m} \frac{\partial}{\partial \bar{m}} + D \right) \Lambda_{\text{CT}} = 0, \quad (19)$$

where $\Delta_{M_H^2} = 2$ and $\Delta_{\Lambda_{\text{CT}}} = D$ are simply the engineering dimensions as M_H^2 and Λ_{CT} are observables which are free from anomalous scaling. Using Eqs. (18), (12), (16) and Eqs. (19), (13), (17) we obtain:

$$\bar{\beta} \left(\frac{\partial}{\partial \bar{g}} M_H^2 + \frac{1}{2\bar{g}} \langle \bar{G} \rangle_{E_H} \right) = \bar{\beta} \left(\frac{\partial}{\partial \bar{g}} \Lambda_{\text{CT}} + \frac{1}{2\bar{g}} \langle \bar{G} \rangle_0 \right) = 0 \quad (20)$$

and from there we read off our main results,

$$\bar{g} \frac{\partial}{\partial \bar{g}} E_H^2 = -\frac{1}{2} \langle \bar{G} \rangle_{E_H}, \quad (21)$$

$$\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{CT}} = -\frac{1}{2} \langle \bar{G} \rangle_0. \quad (22)$$

For the first relation we have used $\frac{\partial}{\partial \bar{g}} M_H^2 = \frac{\partial}{\partial \bar{g}} E_H^2$, where the same remark applies as for the derivative with respect to m given earlier on. In particular this implies that $\langle \bar{G} \rangle_{E_H} = \langle \bar{G} \rangle_{M_H}$ is a static quantity.⁶ The relation remains valid in the case where the quark masses are degenerate as one can replace $m\bar{q}q \rightarrow m_1\bar{q}_1q_1 + m_2\bar{q}_2q_2$ in all relations as well as for the corresponding anomalous mass dimension. By taking the coupling to be dimensionless we have implicitly assumed the space–time dimension to be $D = 4$. The only modification for a derivation in D -dimensions is to replace $\bar{\beta}/2\bar{g} \rightarrow \bar{\beta}/2\bar{g} + (D - 4)/4$. Since $\bar{\beta}$ disappears from the final results (21), (22), the latter are valid for any integer $D \geq 2$. It seems worthwhile to point out that the relations (21), (22) have been checked explicitly [2]; Eq. (21) for the Schwinger model and the Seiberg–Witten theory as well as Eq. (22) for the massive multi flavour Schwinger model. The relations (21), (22) (c.f. also (12), (13)) are akin to Gell-Mann Oakes Renner relations [12] in that they relate an operator expectation value to physical quantities. The scheme dependence of the gluon condensates, inherent in the earlier statement $\bar{G} \neq G$, is made manifest through $\bar{g} \frac{\partial}{\partial \bar{g}} \neq g \frac{\partial}{\partial g}$ and the fact that E_H^2 and Λ_{CT} , being physical quantities, do not renormalise. Thus we wish to stress that it is vital to distinguish bare and renormalised quantities when discussing the relations (21),

(22). The condensates may be computed through (21), (22) with lattice Monte Carlo simulation in some fixed scheme. The conversion to other schemes, say, the $\overline{\text{MS}}$ -scheme can be done through a perturbative computation at some large matching scale. For example defining two schemes a and b through $g = Z_a \bar{g}_a = Z_b \bar{g}_b$ one gets:

$$\bar{g}_a \frac{\partial}{\partial \bar{g}_a} = Z_G^{ab} \bar{g}_b \frac{\partial}{\partial \bar{g}_b}, \quad Z_G^{ab} = \left(1 + \bar{g}_b \frac{\partial}{\partial \bar{g}_b} \ln \frac{Z_a}{Z_b} \right). \quad (23)$$

The transformation between schemes a and b is therefore given by $\langle \bar{G}_a \rangle = Z_G^{ab} \langle \bar{G}_b \rangle$ according to Eqs. (21), (22) for both the vacuum and particle gluon matrix element. The derivation of the relations (21), (22) with bare couplings would surely be possible, but we do not consider it a necessity.

It is worthwhile to illustrate the importance of using eigenstates of the Hamiltonian for the matrix elements considered in the Feynman–Hellmann theorem by an example at hand. One might be tempted to obtain the relation (16) directly from the trace anomaly (10) assuming a mass independent scheme (which entails that $\bar{\beta}$ and $\bar{\gamma}_m$ are independent of m) via

$$\begin{aligned} \bar{m} \frac{\partial}{\partial \bar{m}} M_H^2 &= \bar{m} \frac{\partial}{\partial \bar{m}} \frac{1}{2} \langle T_{\mu}^{\mu} \rangle_{M_H} \\ &= \frac{1}{2} (1 + \bar{\gamma}_m) \langle \bar{Q} \rangle_{M_H} + \text{corrections}, \end{aligned} \quad (24)$$

which without corrections and $\bar{\gamma}_m \neq 1$ contradicts (16). The necessary corrections originate from the fact that T_{μ}^{μ} does not commute with the Hamiltonian in general and therefore is not an eigenoperator of the physical states (7). Thus differentiation of the states with respect to $\bar{m} \frac{\partial}{\partial \bar{m}}$ is required for consistency and exemplifies the importance of the energy eigenstates in the Feynman–Hellmann theorem.

4. Conclusions and discussion

In this work we have derived relations between the logarithmic derivative of the mass of a state (and the vacuum energy) with respect to the gauge coupling in terms of the corresponding gluon condensates as given in Eqs. (21), (22). For the readers convenience we restate the relations within slightly more standard notation:

$$\bar{g} \frac{\partial}{\partial \bar{g}} M_H^2 = -\frac{1}{2} \langle H(E_H) | \frac{1}{\bar{g}^2} \bar{G}^2 | H(E_H) \rangle_c, \quad (25)$$

$$\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{CT}} = -\frac{1}{2} \langle 0 | \frac{1}{\bar{g}^2} \bar{G}^2 | 0 \rangle, \quad (26)$$

where $\frac{\partial}{\partial \bar{g}} M_H^2 = \frac{\partial}{\partial \bar{g}} E_H^2$ as argued earlier on and barred quantities correspond to renormalised quantities. In particular Λ_{CT} and M_H^2 originate from the trace of the energy momentum tensor which is known to be finite after renormalisation of the basic parameters of the theory [7]. Hence the relation above relates finite quantities with each other. We shall comment on the interest of these equations for various aspects in the paragraphs below.

First the $\ln \bar{g}$ -derivative of M_H^2 and Λ_{CT} may be taken as a definition of the gluon condensates. This means that the LHS, computable in lattice Monte Carlo simulations, serves as a definition of the condensates on the RHS. An important point is that by computing the condensates indirectly via derivatives from physical quantities, problems with power divergences, which plague direct approaches, are absent. In this respect our approach constitutes a paradigm shift. For $\langle \bar{G} \rangle_{E_H}$ this should be straightforward since M_H^2 is easily computable whereas for the gluon vacuum condensate $\langle \bar{G} \rangle_0$ this comes with the caveat that Λ_{CT} is not easy to compute

⁴ *En passant* we note that close to the fixed point $\bar{\beta} = 0$, the RGE for Λ_{CT} implies that $\Lambda_{\text{CT}} \sim m^{D/(1+\gamma_m^*)}$ (γ_m^* denotes the fixed point value of γ_m). This observation was essentially already made in our previous paper [10] by deriving $\langle Q \rangle_0 \sim m^{D/(1+\gamma_m^*)}$.

⁵ For pure Yang–Mills (YM) theory, where effectively $Q \rightarrow 0$, we note that (18) leads to $(\bar{\beta}_{\text{YM}})^{-1} = -\frac{\partial}{\partial \bar{g}} \ln M_H$ with H being a glueball state. This may serve as a definition of $\bar{\beta}_{\text{YM}}$. The extension of this idea to a gauge theory with fermions is not immediate.

⁶ We could have obtained this result earlier on by inserting (12) into (16) and using $\langle Q \rangle_{M_H} = \langle Q \rangle_{E_H}$.

by itself. We note that for the former the disconnected part is automatically absent since it does not contribute to the mass M_H^2 . The scheme dependence of the condensate is determined by the scheme dependence of the LHS and has been discussed in the text. The transition from one scheme to another can be achieved by a perturbative computation provided the matching scale is high enough for perturbation theory to be valid.

We shall add a few remarks on the gluon condensates. In QCD the matter condensates $\bar{\beta}/(2\bar{g})\langle\bar{G}\rangle_{E_H}$ are known indirectly through the mass (Eq. (12)) for light mesons, other than the pseudo Goldstone bosons π, K, η, \dots , as for the latter \bar{Q} is negligible since it is $\mathcal{O}(m_{\text{light}})$. For the nucleon this was first discussed in [13]. For the B -meson $\bar{\beta}/(2\bar{g})\langle\bar{G}\rangle_{E_H}$ is related to a non-perturbative definition of the heavy quark scale $\bar{\Lambda}_{\text{HQ}}$ [14]. The determination of the gluon vacuum condensate is of importance for QCD sum rules [15] as well as for the cosmological constant problem to be discussed further below. The value of the gluon condensate cannot be regarded as settled. This is, in part, due to the fact that there is no direct first principle determination of the gluon condensate.

Let us comment on aspects of the cosmological constant, which is a topic of more speculative nature. Without gravity only energy differences matter. Thus the cosmological constant is only determined up to a constant in flat space. Yet the difference of the cosmological constant due to the QCD phase transition itself is generally seen to be a tractable quantity, given by Eq. (13) provided the condensates are well defined. The quark condensate is known through the Gell-Mann Oakes Renner relation [12]: $\langle\bar{Q}\rangle_0 = -f_\pi^2 m_\pi^2 + \mathcal{O}(m)$, with m_π and f_π being the pion mass and decay constants. It would seem that any undetermined constant of the gluon condensate should drop out in Eq. (22). Therefore $\langle\bar{G}\rangle_0$ determined from this equation could be reinserted into Eq. (13), where scheme dependence cancels provided the appropriate $\bar{\beta}$ and \bar{y}_m are used.⁷ Scheme independence in turn might be used as a consistency check of the ideas brought forward in this paragraph.

Let us add that if the gluon condensate can be determined, then it could be checked to what degree the lowest $J^{\text{PC}} = 0^{++}$ -state in a confining gauge theory saturates the partial dilation conserved current hypothesis, see e.g. Ref. [16]. This could serve as a quantitative measure to identify what is commonly referred to as a dilaton in the literature. The possibility that the Higgs boson candidate discovered at the LHC might be a dilaton of a gauge theory with slow running coupling (walking technicolor) is a possibility that is still considered within the particle physics community e.g. [17].

It might be interesting to make use of the relation (25) in approaches where hadron masses can be computed. We are thinking not only of lattice QCD but also of AdS/QCD or Dyson–Schwinger approaches. The gluon condensate could be reinserted, along with the quark condensate, into the trace anomaly (12) and this allows for the extraction of information on the beta function and the anomalous dimension of the mass. In the case where there are either no fermions or fermions with zero mass, the relation in footnote 5 serves as a definition of the beta function of the theory.

Moreover, since the relation applies to any state one can check for the robustness of the results by applying it to many states.

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⁷ A more complete discussion would include the mass degeneracy of the flavours. The effect of heavy flavours ($m_{\text{quark}} > \Lambda_{\text{QCD}}$) can in principle be absorbed into the beta function. The precise discussion of which goes beyond the scope of this Letter.